A_N-type Dunkl operators and new spin Calogero–Sutherland models

F. Finkel,[1] D. Gómez-Ullate,[1] A. González-López,[1] M. A. Rodríguez,[1] and R. Zhdanov[1,*]

[1] *Departamento de Física Teórica II, Universidad Complutense, 28040 Madrid, Spain*

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A new family of A_N-type Dunkl operators preserving a polynomial subspace of finite dimension is constructed. Using a general quadratic combination of these operators and the usual Dunkl operators, several new families of exactly and quasi-exactly solvable quantum spin Calogero–Sutherland models are obtained. These include, in particular, three families of quasi-exactly solvable elliptic spin Hamiltonians.

I. INTRODUCTION

In the early seventies, Calogero [6] and Sutherland [36, 37] introduced the celebrated exactly solvable (ES) and integrable quantum many-body problems in one dimension that bear their names. These papers had a profound impact in the whole physics community, as reflected by the vast amount of literature devoted to the study of the mathematical properties and applications of these models. Among the most recent ones we could mention soliton theory [26, 32], orthogonal polynomials [1, 13, 27], fractional statistics and anyons [8], random matrix theory [39], and Yang–Mills theories [11, 20], to name only a few. Later on, Olshanetsky and Perelomov [29] explained the integrability of the original Calogero–Sutherland (CS) models by relating them to the root system of A_N type. These authors then constructed new families of integrable many-body Hamiltonians associated with all the other root systems. Furthermore, they showed that the most general interaction potential for these models is proportional to the Weierstrass $\wp$ function.

A considerable effort has been devoted over the last decade to the extension of CS models to particles with spin. These models are a step forward towards the unification of the CS scalar models and integrable spin chains, like the Haldane–Shastry model [22, 34]. Several different techniques have been used to construct spin counterparts of the scalar CS models, including the exchange operator method [31], the Dunkl operators formalism [3, 9], the supersymmetric approach [5], and reduction by discrete symmetries [33]. In the quantum case, only the rational and trigonometric (or hyperbolic) spin CS models have been constructed, both in their A_N [2, 3, 4, 23, 25, 31, 38, 42] and BC_N [44] versions. In the A_N case, the integrability and the exact-solvability of these models both follow from the fact that the Hamiltonian is related to a quadratic combination of some family of Dunkl operators.

The Dunkl operators

$$T_i = \frac{\partial}{\partial z_i} + a \sum_{j \neq i} \frac{1}{z_i - z_j}(1 - K_{ij}), \qquad i = 1, \dots, N\,, \tag{1}$$

were originally introduced in [12] in connection with the theory of orthogonal polynomials associated with finite reflection groups. In the latter expression, a is an arbitrary real parameter and the sum runs over $j = 1, \dots, i-1, i+1, \dots, N$. The permutation operators $K_{ij} = K_{ji}$ act on an arbitrary function $f(\mathbf{z})$, with $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{R}^N$, as

$$(K_{ij} f)(z_1, \dots, z_i, \dots, z_j, \dots, z_N) = f(z_1, \dots, z_j, \dots, z_i, \dots, z_N). \tag{2}$$

Using the relations

$$K_{ij}^2 = 1, \quad K_{ij}K_{jk} = K_{ik}K_{ij} = K_{jk}K_{ik}, \quad K_{ij}K_{kl} = K_{kl}K_{ij}, \tag{3}$$

where i, j, k, l take different values in the range $1, \dots, N$, one can establish the commutativity of the operators (1) and prove that $T_i, K_{jk},\ i, j, k = 1, \dots, N$, span a realization of a degenerate

*On leave of absence from Institute of Mathematics, 3 Tereshchenkivska St., 01601 Kyiv - 4 Ukraine

Hecke algebra (see [9] for more details). Since the rational spin CS Hamiltonian is related to a polynomial in the Dunkl operators (1), these operators yield a complete set of commuting integrals of motion. In addition, the spectrum of the Hamiltonian follows immediately from that of the Dunkl operators, which can be easily computed [2, 3].

The previous considerations also apply to the operators

$$\tilde{T}_i = z_i \frac{\partial}{\partial z_i} + a \sum_{j<i} \frac{z_i}{z_i - z_j} (1 - K_{ij}) + a \sum_{j>i} \frac{z_j}{z_i - z_j} (1 - K_{ij}) + 1 - i, \tag{4}$$

$i = 1, \ldots, N$, introduced by Cherednik [9] in connection with the trigonometric spin CS model. In other words, the operators $\tilde{T}_i$ commute, have an easily computable spectrum, and can be used to obtain a complete set of integrals of motion and the spectrum of the Hamiltonian. It has become customary in the literature to refer to both families of operators T_i and $\tilde{T}_i$ as Dunkl operators.

Recently, some partially solvable deformations of the scalar CS models with an external potential have been proposed [16, 24, 28]. For these models —in contrast to the CS models listed in [29]— only a finite-dimensional subset of the spectrum can be computed algebraically. Following Turbiner and Ushveridze [40, 41], we shall use the term *quasi-exactly solvable* (QES) to refer to this type of models; see also the reviews [19, 35, 43]. In all these models, the Hamiltonian can be expressed as a quadratic combination of the generators of a realization of $\mathfrak{sl}(N+1)$ by first-order differential operators preserving a finite-dimensional space of smooth functions. The action of the Hamiltonian in this space can thus be represented by a finite-dimensional constant matrix. The eigenvalues of this matrix belong to the spectrum of the Hamiltonian provided the corresponding eigenfunctions satisfy some appropriate boundary conditions. The Lie algebra $\mathfrak{sl}(N+1)$ is usually referred to as a *hidden symmetry algebra* of the Hamiltonian in these models.

In this paper, we propose a general procedure for constructing (Q)ES spin CS models, close in spirit to the hidden symmetry algebra approach to scalar QES models. The starting point of our construction is the well-known fact that the two standard families of Dunkl operators (1) and (4) admit an infinite sequence of invariant polynomial subspaces of finite dimension. One of the main novelties in our approach consists in the introduction of a new family of commuting Dunkl operators which, together with the other two families (1) and (4), is shown to preserve a single polynomial subspace of finite dimension. We then prove that certain quadratic combinations involving *all* three families of Dunkl operators always yield a spin CS Hamiltonian. The QES character of these Hamiltonians follows immediately from the fact that the Dunkl operators admit a finite-dimensional invariant subspace. Moreover, if the original quadratic combination does not involve the new family of Dunkl operators, the resulting Hamiltonian preserves an infinite sequence of finite-dimensional subspaces of smooth functions, which we shall take as the definition of exact solvability. The linear space spanned by all three types of Dunkl operators is then shown to be invariant under the projective action of the group $\mathrm{GL}(2, \mathbb{R})$. We make use of this fact to perform a complete classification of the resulting (Q)ES spin CS models. All the previously known exactly-solvable spin CS models of A_N type appear as particular cases, arising from a quadratic combination of a *single* type (either (1) or (4)) of Dunkl operators. In addition, we obtain many new spin CS models, both exactly and quasi-exactly solvable. These include, in particular, several elliptic QES spin CS models. To the best of our knowledge, these are the first examples of solvable quantum spin CS models involving elliptic functions.

II. A NEW FAMILY OF A_N-TYPE DUNKL OPERATORS

In this section we shall define a third family of Dunkl operators that preserve certain finite-dimensional polynomial subspaces. These three families shall be used in the following sections to construct exactly and quasi-exactly solvable spin many-body Hamiltonians.

Let us begin by introducing the polynomial subspaces

$$\mathcal{R}_m(\mathbf{z}) = \operatorname{span}\left\{ \prod_{i=1}^{N} z_i^{l_i} \ : \ l_i \leq m, \quad i = 1, \ldots, N \right\}, \tag{5}$$

$$\mathcal{P}_n(\mathbf{z}) = \operatorname{span}\left\{ \prod_{i=1}^{N} z_i^{l_i} \ : \ \sum_{i=1}^{N} l_i \leq n \right\}, \tag{6}$$

which shall be referred to as the *rectangular* and *triangular* modules, respectively, by analogy with the two particle case [15].

A well-known property of the Dunkl operators (1), (4) is the fact that they preserve the triangular module $\mathcal{P}_n(\mathbf{z})$ for arbitrary n. Seemingly less known, but central to our construction, is the fact that they preserve the rectangular module $\mathcal{R}_m(\mathbf{z})$ for arbitrary m as well. On the other hand, the differential parts of the Dunkl operators (1), (4) together with the differential operator $z_i^2\partial_{z_i}$, span a realization of $\mathfrak{sl}(2)$. Inspired by this fact, it is natural to suggest the following ansatz for a third set of Dunkl operators:

$$J_i = z_i^2\frac{\partial}{\partial z_i} - m z_i + \sum_{j\neq i} f_{ij}(\mathbf{z})(1-K_{ij}), \qquad i=1,\dots,N,$$

where m is an arbitrary non-negative integer and $f_{ij}(\mathbf{z})$ is a function anti-symmetric in i,j. This new family does not preserve the module $\mathcal{P}_n(\mathbf{z})$, but for a suitable choice of the functions $f_{ij}(\mathbf{z})$ it will be shown to preserve the module $\mathcal{R}_m(\mathbf{z})$.

To this end, let us define the operators

$$\begin{aligned} Q_i^- &= a\sum_{j\neq i}\frac{1}{z_i-z_j}(1-K_{ij}),\\ Q_i^0 &= \frac{a}{2}\sum_{j\neq i}\frac{z_i+z_j}{z_i-z_j}(1-K_{ij}),\\ Q_i^+ &= a\sum_{j\neq i}\frac{z_iz_j}{z_i-z_j}(1-K_{ij}), \end{aligned} \tag{7}$$

where a is a real parameter and the sum runs over $j=1,\dots i-1,i+1,\dots N$.

The following lemma will be important in the sequel:

Lemma 1. *For any non-negative integer n, the rectangular module $\mathcal{R}_n(\mathbf{z})$ is invariant under the action of the operators Q_i^-, Q_i^0 and Q_i^+. The triangular module $\mathcal{P}_n(\mathbf{z})$ is invariant only under the action of the operators Q_i^- and Q_i^0.*

Proof. It suffices to prove that the inclusions

$$\begin{aligned} &\frac{h_{ij}^\epsilon}{z_i-z_j}(1-K_{ij})\,\mathcal{R}_n(\mathbf{z})\subset\mathcal{R}_n(\mathbf{z})\,, \quad \epsilon=\pm,0,\\ &\frac{h_{ij}^\epsilon}{z_i-z_j}(1-K_{ij})\,\mathcal{P}_n(\mathbf{z})\subset\mathcal{P}_n(\mathbf{z})\,, \quad \epsilon=-,0, \end{aligned}$$

hold for any pair of indices $i\neq j$, where

$$h_{ij}^- = 1\,, \qquad h_{ij}^0 = z_i+z_j\,, \qquad h_{ij}^+ = z_iz_j.$$

The action of these operators on an arbitrary monomial

$$\frac{h_{ij}^\epsilon}{z_i-z_j}(1-K_{ij})\prod_{k=1}^N z_k^{l_k} = \frac{z_i^{l_i}z_j^{l_j}-z_i^{l_j}z_j^{l_i}}{z_i-z_j}\,\frac{h_{ij}^\epsilon}{z_i^{l_i}z_j^{l_j}}\prod_{k=1}^N z_k^{l_k}$$

yields a polynomial, since $z_i^{l_i}z_j^{l_j}-z_i^{l_j}z_j^{l_i}$ is a multiple of z_i-z_j. The homogeneous degree of this polynomial is either 0 (if $l_i=l_j$) or $\epsilon+\sum_i l_i$ (if $l_i\neq l_j$). Therefore, if the original monomial belongs to $\mathcal{P}_n(\mathbf{z})$ the resulting polynomial is also in $\mathcal{P}_n(\mathbf{z})$ for $\epsilon=-,0$, but lies outside $\mathcal{P}_n(\mathbf{z})$ for $\epsilon=+$.

On the other hand, the degrees of the variables z_k in the resulting polynomial remain equal to l_k if $k\neq i,j$, while the degrees of z_i and z_j satisfy

$$\deg(z_k)\le\max(l_i,l_j)-1+d_\epsilon\,, \quad k=i,j\,,$$

where

$$d_\epsilon=\begin{cases}0 & \text{if } \epsilon=-,\\ 1 & \text{if } \epsilon=0,+.\end{cases}$$

Therefore, if the original monomial belongs to the space $\mathcal{R}_n(\mathbf{z})$ so does the resulting polynomial in all three cases $\epsilon = \pm, 0$. $\square$ $\square$

The following three sets of Dunkl operators shall be the building blocks for the construction of several new (quasi-)exactly solvable spin CS models:

$$\begin{aligned} J_i^- &= \frac{\partial}{\partial z_i} + a \sum_{j \neq i} \frac{1}{z_i - z_j}(1 - K_{ij}), \\ J_i^0 &= z_i \frac{\partial}{\partial z_i} - \frac{m}{2} + \frac{a}{2} \sum_{j \neq i} \frac{z_i + z_j}{z_i - z_j}(1 - K_{ij}), \\ J_i^+ &= z_i^2 \frac{\partial}{\partial z_i} - m z_i + a \sum_{j \neq i} \frac{z_i z_j}{z_i - z_j}(1 - K_{ij}), \end{aligned} \tag{8}$$

where $i = 1, \dots, N$, a is a real parameter, and m is a non-negative integer. Note that the operators J_i^- coincide exactly with the Dunkl operators (1), while the operators J_i^0 differ from the Cherednik operators (4) by a linear combination with constant coefficients of the permutation operators K_{ij}, namely

$$\tilde{T}_i = J_i^0 + \frac{a}{2} \sum_{j<i} (1 - K_{ij}) - \frac{a}{2} \sum_{j>i} (1 - K_{ij}) + \frac{m}{2} + 1 - i\,. \tag{9}$$

The operators J_i^+ are, to the best of our knowledge, new.

The operators J_i^ϵ and K_{ij} obey the following commutation relations

$$[J_i^\pm, J_j^\pm] = 0\,, \qquad [J_i^0, J_j^0] = \frac{a^2}{4} \sum_{k \neq i,j} K_{ij}(K_{jk} - K_{ik})\,, \tag{10}$$

$$[K_{ij}, J_k^\epsilon] = 0\,, \qquad K_{ij} J_i^\epsilon = J_j^\epsilon K_{ij}\,, \tag{11}$$

where $\epsilon = \pm, 0$ and the indices i, j, k are all different. The set of operators

$$\{J_i^\epsilon, K_{ij} \;:\; i, j = 1, \dots, N\}\,, \qquad \epsilon = \pm, 0,$$

spans a degenerate affine Hecke algebra, see [9]. This is clear for $\epsilon = \pm$, while for $\epsilon = 0$, it follows from (9) and the commutativity of the Cherednik operators (4).

The key property in our construction of (quasi-)exactly solvable spin CS models is the fact that the operators (8) possess invariant polynomial subspaces.

Theorem 1. *The operators J_i^- and J_i^0 preserve the modules $\mathcal{P}_n(\mathbf{z})$ and $\mathcal{R}_n(\mathbf{z})$ for an arbitrary non-negative integer n. The operators J_i^+ preserve the module $\mathcal{R}_m(\mathbf{z})$, but do not preserve the modules $\mathcal{P}_n(\mathbf{z})$ and $\mathcal{R}_k(\mathbf{z})$ for $k \neq m$.*

Proof. The statement follows from Lemma 1 and the fact that the differential parts of J_i^- and J_j^0 preserve the modules $\mathcal{P}_n(\mathbf{z})$ and $\mathcal{R}_n(\mathbf{z})$ for any non-negative integer n, whereas the differential part of J_i^+ preserves the module $\mathcal{R}_k(\mathbf{z})$ only for $k = m$. $\square$ $\square$

The following corollary is an immediate consequence of Theorem 1:

Corollary 1. *Any polynomial in the operators J_i^ϵ leaves invariant the rectangular module $\mathcal{R}_m(\mathbf{z})$. In addition, if the polynomial does not depend on J_i^+, it preserves the modules $\mathcal{R}_n(\mathbf{z})$ and $\mathcal{P}_n(\mathbf{z})$ for all n.*

III. CONSTRUCTION OF SPIN CALOGERO–SUTHERLAND MODELS

In the previous section we have introduced a new set of Dunkl operators preserving the space of polynomials $\mathcal{R}_m$. Here we shall make use of all three sets of Dunkl operators (8) to construct some multi-parameter families of spin CS models.

Consider the spin permutation operators S_{ij}, $i,j=1,\dots,N$, whose action on a spin state $|s_1,\dots,s_N\rangle$, $-M\le s_i\le M$, with $M\in\frac12\mathbb{N}$, is given by

$$S_{ij}|s_1,\dots,s_i,\dots,s_j,\dots,s_N\rangle=|s_1,\dots,s_j,\dots,s_i,\dots,s_N\rangle\,. \tag{12}$$

Note that the operators S_{ij} obey the identities (3) with K_{ij} replaced by S_{ij}. Let $\mathfrak{S}$ denote the linear space span$\big\{\,|s_1,\dots,s_N\rangle\big\}_{-M\le s_i\le M}$. The action of the operators S_{ij} in $\mathfrak{S}$ is thus represented by $(2M+1)^N$-dimensional symmetric matrices.

The starting point of our procedure is the following quadratic combination of the Dunkl operators (8):

$$\begin{aligned}-H^*=\sum_i\Big(&c_{++}(J_i^+)^2+c_{00}(J_i^0)^2+c_{--}(J_i^-)^2+\frac{c_{0+}}{2}\{J_i^0,J_i^+\}+\frac{c_{0-}}{2}\{J_i^0,J_i^-\}\\&+c_+J_i^++c_0J_i^0+c_-J_i^-\Big),\end{aligned}\tag{13}$$

where $c_{\epsilon\epsilon'},c_\epsilon\,,\ \epsilon,\epsilon'=\pm,0\,,$ are arbitrary real constants. The term $\frac12\sum_i\{J_i^-,J_i^+\}$ differs from $\sum_i(J_i^0)^2$ by a constant operator (see Appendix), and for this reason it has not been included in (13). We emphasize that only the particular cases of (13)

$$-H^*=c_{00}\sum_i(J_i^0)^2\,,\qquad -H^*=c_{--}\sum_i(J_i^-)^2$$

have been previously discussed in the literature in connection with CS models; see [3, 9, 13, 31, 44] and references therein.

As it is customary, we shall identify K_{ij}, S_{ij}, and H^* with their natural extensions $K_{ij}\otimes 1\!\!\mathrm{I}$, $1\!\!\mathrm{I}\otimes S_{ij}$, and $H^*\otimes 1\!\!\mathrm{I}$ to the tensor product $\mathbb{C}[z_1,\dots,z_N]\otimes\mathfrak{S}$. The following lemma is an immediate consequence of Eq. (11).

Lemma 2. *The (Q)ES differential-difference operator H^* commutes with K_{ij} and S_{ij} for all $i,j=1,\dots,N$.*

This property plays a crucial role in the construction of spin CS models; see, for instance, Ref. [2].

Let Λ be the projection operator on states antisymmetric under the simultaneous interchange of any two particles' coordinates and spins. In terms of the total permutation operators $\Pi_{ij}=K_{ij}S_{ij}$, the operator Λ can be alternatively defined by the relations $\Pi_{ij}\Lambda=-\Lambda$, $j>i=1,\dots,N$. Since $K_{ij}^2=1$, these relations are equivalent to

$$K_{ij}\Lambda=-S_{ij}\Lambda,\qquad j>i=1,\dots,N. \tag{14}$$

For the lowest values of N the antisymmetrizer Λ is given by

$$\begin{aligned}&N=2: &&\Lambda=1-\Pi_{12}\,,\\&N=3: &&\Lambda=1-\Pi_{12}-\Pi_{13}-\Pi_{23}+\Pi_{12}\Pi_{13}+\Pi_{12}\Pi_{23}\,.\end{aligned}$$

In general, Λ is an $(N-1)$-th degree polynomial in the total permutation operators Π_{ij}. It thus follows from Lemma 2 that H^* commutes with Λ.

Suppose that $f(\mathbf{z})$ is an eigenfunction of H^* with eigenvalue λ. For instance, f could be one of the polynomial eigenfunctions that H^* is guaranteed to possess in $\mathcal{R}_m$. Given any (constant) spin state $|\sigma\rangle\in\mathfrak{S}$, the spin function $\varphi=\Lambda[f(\mathbf{z})|\sigma\rangle]$ is also an eigenfunction of H^* with the same eigenvalue λ.

Next, we introduce the matrix differential operator $\overline{H}$ obtained from H^* by the formal substitutions $K_{ij}\to -S_{ij}$, $i,j=1,\dots,N$. The relations (14) imply that φ is a spin eigenfunction of $\overline{H}$ with eigenvalue λ.

Using the formulae (A1)–(A6) for the sums of the squares and the anticommutators of the Dunkl operators (8) given in the Appendix, we obtain the following explicit expression for $\overline{H}$:

$$\begin{aligned}-\overline{H}=&\sum_i\Big(P(z_i)\partial_{z_i}^2+\tilde Q(z_i)\partial_{z_i}+R(z_i)\Big)+2ac_{++}(1-m)\sum_{i<j}z_iz_j\\&+2a\sum_{i<j}\frac{1}{z_i-z_j}\big(P(z_i)\partial_{z_i}-P(z_j)\partial_{z_j}\big)-a\sum_{i<j}\frac{P(z_i)+P(z_j)}{(z_i-z_j)^2}\,(1+S_{ij})\\&+a\sum_{i<j}\big(c_{++}(z_i+z_j)^2+c_{0+}(z_i+z_j)+c_{00}\big)\,S_{ij}+\frac{a^2}{12}\,c_{00}{\sum_{i,j,k}}'(1-S_{ij}S_{ik}),\end{aligned}\tag{15}$$

where

$$
\begin{aligned}
P(z) &= c_{++}z^4 + c_{0+}z^3 + c_{00}z^2 + c_{0-}z + c_{--}\,,\\
\tilde{Q}(z) &= Q(z) + \Big(1-\frac{b}{2}\Big)\,P'(z)\,,\\
Q(z) &= c_+z^2 + c_0 z + c_-\,, \qquad b = 1+m+a(N-1)\,,\\
R(z) &= c_{++}\big(b+m(m-2)-1\big)z^2 + \Big[c_{0+}\Big[\Big(1-\frac{m}{2}\Big)b + m(m-1)-1\Big] - mc_+\Big]z\\
&\quad + \frac{c_{00}}{4}\,\big(2(b-1)+m(m-2)\big) - \frac{m}{2}\,c_0\,,
\end{aligned}
\tag{16}
$$

and $\sum'_{i,j,k}$ denotes summation in i,j,k with $i\neq j\neq k\neq i$.

In the final step, one performs a gauge transformation with a suitable scalar function $\mu(\mathbf{z})$, followed by a change of variables $\mathbf{z}=\boldsymbol{\zeta}(\mathbf{x})$, $\mathbf{x}=(x_1,\dots,x_N)$,

$$
H = \mu\,\overline{H}\,\mu^{-1}\Big|_{\mathbf{z}=\boldsymbol{\zeta}(\mathbf{x})}\,, \tag{17}
$$

in order to reduce the *gauge spin Hamiltonian* $\overline{H}$ to the *Schrödinger form*

$$
H = -\sum_i \partial_{x_i}^2 + V(\mathbf{x})\,, \tag{18}
$$

where $V(\mathbf{x})$ is a Hermitian matrix-valued function.

Proposition 1. *The operator $\overline{H}$ in Eq.* (15) *can be reduced to a matrix Schrödinger operator* (18) *by a change of variables $\mathbf{z}=\boldsymbol{\zeta}(\mathbf{x})$ and conjugation by a scalar gauge factor $\mu(\mathbf{z})$.*

Proof. The gauge transformation (17) with gauge factor

$$
\mu(\mathbf{z}) = \prod_{i<j}(z_i-z_j)^a\,\prod_{i=1}^N P(z_i)^{-\frac14}\,\exp\int^{z_i}\frac{\tilde{Q}(y)}{2P(y)}\,\mathrm{d}y, \tag{19}
$$

together with the change of variables

$$
x_i = \zeta^{-1}(z_i) = \int^{z_i}\frac{\mathrm{d}y}{\sqrt{P(y)}}\,, \qquad i=1,\dots,N, \tag{20}
$$

map the gauge spin Hamiltonian $\overline{H}$ to a matrix Schrödinger operator (18), with potential

$$
\begin{aligned}
V(\mathbf{x}) &= a^2{\sum_{i,j,k}}'\frac{P(z_i)}{(z_i-z_j)(z_i-z_k)} + a\sum_{i\neq j}\frac{\tilde{Q}(z_i)}{z_i-z_j} + a\sum_{i<j}\frac{P(z_i)+P(z_j)}{(z_i-z_j)^2}\,(a+S_{ij})\\
&\quad - 2ac_{++}(1-m)\sum_{i<j}z_iz_j - a\sum_{i<j}\big(c_{++}(z_i+z_j)^2 + c_{+0}(z_i+z_j)+c_{00}\big)\,S_{ij}\\
&\quad + \frac14\sum_i\Big[\frac{1}{P(z_i)}\Big(\frac34 P'(z_i)^2 + \tilde{Q}(z_i)^2 - 2\tilde{Q}(z_i)P'(z_i)\Big) - P''(z_i) + 2\tilde{Q}'(z_i)\\
&\quad - 4R(z_i)\Big] - \frac{a^2}{12}\,c_{00}{\sum_{i,j,k}}'(1-S_{ij}S_{ik})\Bigg|_{\mathbf{z}=\boldsymbol{\zeta}(\mathbf{x})}\,. \qquad \square
\end{aligned}
\tag{21}
$$

$\square$

Remark 1. The gauge factor $\mu(\mathbf{z})$ in (19) was introduced in [16] in connection with a generalization of the theory of QES models with one degree of freedom to many-body problems. The existence of a (matrix or scalar) gauge factor and change of coordinates reducing a given matrix second-order differential operator in $N>1$ variables to a matrix Schrödinger operator (18) is not guaranteed a priori. In the scalar case this problem was first addressed by Cotton [10], while the matrix case

has been studied recently in Ref. [14]. In fact, the quadratic combination H^* has been chosen so that a scalar gauge factor and change of variables can be easily found for $\overline{H}$. For instance, the term $\sum_i [J_i^+, J_i^-]$ has been discarded because it involves first-order derivatives with matrix-valued coefficients, which are usually very difficult to gauge away.

Remark 2. The change of variables (20), and hence the potential $V(\mathbf{x})$, are defined up to an arbitrary translation for each coordinate x_i, $i = 1, \dots, N$. We shall see that this arbitrariness can be removed in some cases by requiring the potential to be invariant under sign reversals of any coordinate x_i.

If $\varphi(\mathbf{z})$ is one of the eigenfunctions of $\overline{H}$ with eigenvalue λ constructed above, the spin function $\psi(\mathbf{x}) = \mu\big(\boldsymbol{\zeta}(\mathbf{x})\big)\varphi\big(\boldsymbol{\zeta}(\mathbf{x})\big)$ is clearly an eigenfunction of H with the same eigenvalue. Note, however, that we have not imposed so far any boundary conditions on the eigenfunctions ψ. In general, the parameters a, $c_{\epsilon\epsilon'}$, and c_ϵ defining H^* should satisfy certain constraints in order to ensure that the appropriate boundary conditions are satisfied.

The following proposition is an immediate consequence of the previous considerations:

Proposition 2. *The spin Schrödinger operator* (18) *with potential* (21) *leaves invariant the module*

$$\mathcal{M}_m = \mu\big(\boldsymbol{\zeta}(\mathbf{x})\big)\Lambda\Big(\mathcal{R}_m\big(\boldsymbol{\zeta}(\mathbf{x})\big) \otimes \mathfrak{S}\Big). \tag{22}$$

In addition, if $c_{++} = c_{0+} = c_+ = 0$, it preserves the modules $\mathcal{M}_n$ and

$$\mathcal{N}_n = \mu\big(\boldsymbol{\zeta}(\mathbf{x})\big)\Lambda\Big(\mathcal{P}_n\big(\boldsymbol{\zeta}(\mathbf{x})\big) \otimes \mathfrak{S}\Big), \tag{23}$$

for any non-negative integer n.

If we add a constant term

$$V_0 = \gamma_0 + \gamma_1 \sum_{i<j} S_{ij} + \gamma_2 {\sum_{i,j,k}}' S_{ij} S_{ik}\,, \qquad \gamma_i \in \mathbb{R}, \tag{24}$$

to the potential (21), the previous procedure for constructing eigenfunctions of H still applies to $H + V_0$. Indeed, the associated operator $(H + V_0)^*$ is obtained by adding the term

$$V_0^* = \gamma_0 - \gamma_1 \sum_{i<j} K_{ij} + \gamma_2 {\sum_{i,j,k}}' K_{ij} K_{ik}$$

to the initial operator H^*. Our assertion follows from the fact that V_0^* preserves the modules $\mathcal{P}_n$ and $\mathcal{R}_n$ for all n, commutes with all the permutation operators K_{ij}, and acts trivially on $\mathfrak{S}$. This observation will be used in what follows to simplify the formula for $V(\mathbf{x})$ by dropping terms of the form (24).

IV. CLASSIFICATION OF SPIN CALOGERO–SUTHERLAND MODELS

We have seen in Section III that any quadratic combination of the form (13) yields a spin CS model for which a number of eigenvalues and their corresponding eigenfunctions can be computed in an algebraic fashion. In this section we shall obtain a complete classification of the potentials constructed in this way.

The form of the potential $V(\mathbf{x})$ in Eq. (21) depends on the choice of parameters $c_{\epsilon\epsilon'}$ and c_ϵ, $\epsilon, \epsilon' = \pm, 0$. The parameters $c_{\epsilon\epsilon'}$ which define the polynomial P are of particular significance, since they determine the form of the change of variables (20). However, different sets of parameters $c_{\epsilon\epsilon'}$, c_ϵ defining the operator H^* may give rise to the same potential. Indeed, there is a group of residual transformations preserving the vector spaces $\operatorname{span}\{J_i^-, J_i^0, J_i^+\}$, $i = 1, \dots, N$. The image of the operator H^* under these transformations is still of the form (13), albeit with different coefficients $\hat{c}_{\epsilon\epsilon'}$ and $\hat{c}_\epsilon$. We shall make use of this fact to classify the starting multi-parameter family of operators (13) into conjugacy classes. This will provide a complete classification of the

spin CS models obtainable within our framework. The ideas used for this classification are similar in spirit to those applied in Refs. [18, 19] to classify one-particle Lie-algebraic QES Schrödinger operators.

Consider the mapping

$$J_i^\epsilon(\mathbf{w}) \mapsto \widehat{J}_i^\epsilon(\mathbf{z}) = \mu_m(\mathbf{z})\, J_i^\epsilon\big(\mathbf{w}(\mathbf{z})\big)\, \mu_m^{-1}(\mathbf{z})\,, \tag{25}$$

where $\mathbf{w} = (w_1, \ldots, w_N)$ is given by the projective action of $\mathrm{GL}(2,\mathbb{R})$ on $\mathbb{RP}^1$ (Möbius transformation)

$$w_i = \frac{\alpha z_i + \beta}{\gamma z_i + \delta}\,, \qquad i = 1, \ldots, N, \quad \Delta = \alpha\delta - \beta\gamma \neq 0, \tag{26}$$

and the gauge factor $\mu_m(\mathbf{z})$ is defined by

$$\mu_m(\mathbf{z}) = \prod_{i=1}^N (\gamma\, z_i + \delta)^m. \tag{27}$$

The following lemma is an immediate consequence of the definition of the Dunkl operators (8):

Lemma 3. *The mapping* (25) *acts linearly on the vector spaces* $\operatorname{span}\{J_i^-, J_i^0, J_i^+\}$, $i = 1, \ldots, N$, *as*

$$\begin{pmatrix} \widehat{J}_i^+(\mathbf{z}) \\ \widehat{J}_i^0(\mathbf{z}) \\ \widehat{J}_i^-(\mathbf{z}) \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta + \beta\gamma & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix} \begin{pmatrix} J_i^+(\mathbf{z}) \\ J_i^0(\mathbf{z}) \\ J_i^-(\mathbf{z}) \end{pmatrix}. \tag{28}$$

It follows from the previous lemma that the operator $\widehat{H}^*$ defined by

$$\widehat{H}^* = \mu_m(\mathbf{z})\, H^*\big(\mathbf{w}(\mathbf{z})\big)\, \mu_m^{-1}(\mathbf{z})$$

is still a second degree polynomial in the Dunkl operators $J_i^\epsilon(\mathbf{z})$, whose coefficients $\hat{c}_{\epsilon\epsilon'}$, $\hat{c}_\epsilon$ can be easily computed using Eq. (28). The corresponding gauge spin Hamiltonian $\widehat{\overline{H}}$ can be obtained from Eq. (15) by replacing the coefficients $c_{\epsilon\epsilon'}$ and c_ϵ by their counterparts $\hat{c}_{\epsilon\epsilon'}$ and $\hat{c}_\epsilon$. In particular, the polynomials $P(z)$ and $Q(z)$ in Eq. (16) are replaced by the polynomials

$$\widehat{P}(z) = \hat{c}_{++}z^4 + \hat{c}_{0+}z^3 + \hat{c}_{00}z^2 + \hat{c}_{0-}z + \hat{c}_{--}\,, \qquad \widehat{Q}(z) = \hat{c}_+z^2 + \hat{c}_0 z + \hat{c}_-\,. \tag{29}$$

Expressing the coefficients $\hat{c}_{\epsilon\epsilon'}$ and $\hat{c}_\epsilon$ in terms of the original coefficients $c_{\epsilon\epsilon'}$ and c_ϵ, one easily arrives to the explicit formulas

$$\widehat{P}(z) = \frac{(\gamma z + \delta)^4}{\Delta^2}\, P\bigg(\frac{\alpha z + \beta}{\gamma z + \delta}\bigg)\,, \qquad \widehat{Q}(z) = \frac{(\gamma z + \delta)^2}{\Delta}\, Q\bigg(\frac{\alpha z + \beta}{\gamma z + \delta}\bigg)\,. \tag{30}$$

Recall [30] that the (irreducible) multiplier representation $\rho_{n,i}$ of $\mathrm{GL}(2,\mathbb{R})$ on the space of univariate polynomials of degree at most n is defined by the linear transformations

$$p(z) \mapsto \hat{p}(z) = \Delta^i(\gamma z + \delta)^n\, p\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right).$$

By Eq. (30), the polynomials P and Q defining $\overline{H}$ transform according to the representations $\rho_{4,-2}$ and $\rho_{2,-1}$, respectively. Note also that the Dunkl operators (8) transform according to the representation $\rho_{2,-1}$; see Eq. (28). The (nonzero) orbits of the representation $\rho_{4,-2}$ can be parametrized by the following canonical forms [19, 21]:

$$\begin{gathered} \pm 1\,, \quad z\,, \quad \pm\nu(1 - z^2)\,, \quad \pm\nu(1 + z^2)\,, \quad \pm\nu z^2\,, \quad \pm\nu(1 + z^2)^2\,, \\ \pm\nu(1 - z^2)(1 - k^2z^2)\,, \quad \pm\nu(1 + z^2)\big(1 + k^2z^2\big)\,, \quad \nu(1 - z^2)\big(1 - k^2 + k^2z^2\big)\,, \end{gathered}$$

where $\nu > 0$ and $0 < k < 1$. The above list of canonical forms can be further reduced without any loss of generality using the *complex* projective (linear) transformation $w = \mathrm{i}z$. Since the projective

transformation $w = \mathrm{i}z$ also induces the mapping $c_\epsilon \mapsto \hat{c}_\epsilon = \mathrm{i}^\epsilon c_\epsilon$ for the coefficients of the polynomial Q, the resulting canonical form leads to a real potential provided that the initial coefficients $c_\pm$ are purely imaginary. The reduced list of canonical forms will be conveniently taken as

$$\begin{array}{llll} 1) & 1\,, & 6) & \nu(1+z^2)^2\,,\\ 2) & z\,, & 7) & \nu(1-z^2)(1-k^2z^2)\,,\\ 3) & \nu(z^2-1)\,, & 8) & \nu(z^2-1)\big(1-k'^2z^2\big)\,,\\ 4) & \nu(1-z^2)\,, & 9) & \nu(1-z^2)\big(k'^2+k^2z^2\big)\,,\\ 5) & \nu z^2\,, & & \end{array} \tag{31}$$

where $k'^2 = 1-k^2$. By choosing $P(z)$ in Eqs. (15), (16) in each of the canonical forms (31), one obtains a complete classification of the spin CS models with potential (21), which are (Q)ES by construction. Recall that in the one-particle case the canonical forms 1), 2), 3), 5) give rise to rational or hyperbolic potentials, while the remaining five yield periodic (trigonometric or elliptic) potentials [19].

Remark 3. The operator H^* in (13) is easily seen to preserve the module $\mathcal{S}_m(\mathbf{z})$ of symmetric polynomials in $z_1,\dots,z_N$ of degree at most m, on which the permutation operators K_{ij} act as the identity. Hence the antisymmetrizer Λ acts on the space $\mathcal{S}_m\otimes\mathfrak{S}$ as the tensor product $1\!\!1\otimes\Lambda_0$, where Λ_0 is the spin antisymmetrization operator, and the H-invariant module $\mu\Lambda\big(\mathcal{S}_m\otimes\mathfrak{S}\big)$ factors as the tensor product $(\mu\,\mathcal{S}_m)\otimes(\Lambda_0\mathfrak{S})$. The spin permutation operators S_{ij} on this module reduce to $-1\!\!1$. Therefore, the restriction of the Hamiltonian H to this space is simply $\big(H\big|_{S_{ij}\to-1}\big)\otimes 1\!\!1$. Thus the *scalar* Schrödinger operator $H\big|_{S_{ij}\to-1}$ leaves invariant the module $\mu\,\mathcal{S}_m$. It follows that replacing the spin permutation operators S_{ij} by -1 in one of the (Q)ES spin potentials listed below, one obtains a corresponding (Q)ES scalar potential. The scalar potentials so constructed include as particular cases all the potentials presented in [16].

For each of the canonical forms (31), the potential turns out to decompose as

$$V(\mathbf{x}) = \sum_i U(x_i) + V_{\mathrm{int}}(\mathbf{x})\,, \tag{32}$$

where U plays the role of a external field potential, and the interaction potential V_{int} is of the form

$$V_{\mathrm{int}}(\mathbf{x}) = \sum_{i<j}\big(V^-(x_i-x_j)+V^+(x_i+x_j)\big)\,a(a+S_{ij})\,, \tag{33}$$

with either $V^+=0$ or $V^+=V^-$. Indeed, making use of the identities (A7)–(A15) in the Appendix and recalling that $c_{0+}=0$ for all the canonical forms, the potential (21) reduces after some algebra to the form (32) with

$$\begin{aligned} U &= c_{++}(1-b^2)z^2 + b\,c_+z + \frac{1}{16\,P(z)}\Big(3P'(z)^2+4\tilde{Q}(z)^2-8\tilde{Q}(z)P'(z)\Big)\,,\\ V_{\mathrm{int}} &= 2\sum_{i<j}(z_i-z_j)^{-2}\Big(c_{++}z_i^2z_j^2+c_{00}z_iz_j+\frac{c_{0-}}{2}\,(z_i+z_j)+c_{--}\Big)\,a(a+S_{ij})\,. \end{aligned}$$

We have discarded here a constant term of the form (24), in accordance with the observation at the end of Section III. The interaction potential takes the form (33) after performing the change of variables (20) and using some identities for the corresponding function $\zeta(x)$. Moreover, if the function V^+ is nonzero, it can be reduced to V^- by a suitable coordinate translation (see Remark 3).

We shall now present the list of (Q)ES spin many-body potentials obtained from the canonical forms (31). Note that the scaling $(c_{\epsilon\epsilon'},c_\epsilon)\mapsto(\lambda c_{\epsilon\epsilon'},\lambda c_\epsilon)$ induces the mapping

$$\begin{aligned} V(\mathbf{x}\,;c_{\epsilon\epsilon'},c_\epsilon) &\mapsto V(\mathbf{x}\,;\lambda\,c_{\epsilon\epsilon'},\lambda\,c_\epsilon) = \lambda\,V(\sqrt{\lambda}\,\mathbf{x}\,;c_{\epsilon\epsilon'},c_\epsilon)\\ \mu(\mathbf{x}\,;c_{\epsilon\epsilon'},c_\epsilon) &\mapsto \mu(\mathbf{x}\,;\lambda\,c_{\epsilon\epsilon'},\lambda\,c_\epsilon) \propto \mu(\sqrt{\lambda}\,\mathbf{x}\,;c_{\epsilon\epsilon'},c_\epsilon) \end{aligned}$$

of the corresponding potentials and gauge factors. For this reason we shall list the potentials for a suitably chosen value of the parameter ν in Cases 3–9, or a suitable multiple of $P(z)$ in Cases 1,2. The notation

$$x_{ij}^{\pm} = x_i \pm x_j\,; \qquad \alpha_\epsilon = \frac{c_\epsilon}{4}\,, \quad \epsilon = \pm, 0\,; \qquad \alpha = \alpha_+ + \alpha_0 + \alpha_-$$

shall be employed in what follows.

Case 1. $P(z) = \dfrac{1}{4}$.
Change of variables: $z = \dfrac{x}{2}$.
Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} (x_{ij}^-)^a \prod_i \exp\biggl(\frac{1}{3}\,\alpha_+ x_i^3 + \alpha_0 x_i^2 + 4\alpha_- x_i\biggr).$$

External potential:

$$U(x) = \alpha_+^2 x^4 + 4\alpha_0\alpha_+ x^3 + 4(\alpha_0^2 + 2\alpha_-\alpha_+)\,x^2 + 2(8\alpha_-\alpha_0 + b\alpha_+)\,x\,.$$

Interaction potential:

$$V_{\rm int}(\mathbf{x}) = 2\sum_{i<j} (x_{ij}^-)^{-2}\, a(a + S_{ij})\,.$$

Case 2. $P(z) = 4z$.
Change of variables: $z = x^2$.
Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \left(x_{ij}^-\, x_{ij}^+\right)^a \prod_i x_i^{\frac{1}{2}(1-b)+\alpha_-}\, \exp\biggl(\frac{1}{4}\,\alpha_+ x_i^4 + \frac{1}{2}\,\alpha_0 x_i^2\biggr).$$

External potential:

$$U(x) = \alpha_+^2 x^6 + 2\alpha_0\alpha_+ x^4 + \bigl(\alpha_0^2 + \alpha_+(3b + 2\alpha_-)\bigr)x^2 + \frac{1}{4}\,\bigl((2\alpha_- - b)^2 - 1\bigr)\,\frac{1}{x^2}\,.$$

Interaction potential:

$$V_{\rm int}(\mathbf{x}) = 2\sum_{i<j} \Bigl(\bigl(x_{ij}^-\bigr)^{-2} + \bigl(x_{ij}^+\bigr)^{-2}\Bigr)\, a(a + S_{ij})\,.$$

Case 3. $P(z) = 4(z^2 - 1)$.
Change of variables: $z = \cosh 2x$.
Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \bigl(\sinh x_{ij}^-\, \sinh x_{ij}^+\bigr)^a \prod_i (\sinh x_i)^{\frac{1}{2}(1+\alpha-b)} (\cosh x_i)^{\frac{1}{2}(1+2\alpha_0-\alpha-b)} \exp\Bigl(\frac{\alpha_+}{2}\cosh 2x_i\Bigr).$$

External potential:

$$\begin{aligned} U(x) = {} & \alpha_+^2\cosh^2 2x + 2\alpha_+(\alpha_0 + b)\cosh 2x + 2(\alpha_+ + \alpha_-)(\alpha_0 - b)\cosh 2x\,\sinh^{-2}2x \\ & + \bigl((\alpha_+ + \alpha_-)^2 + (\alpha_0 - b)^2 - 1\bigr)\,\sinh^{-2}2x\,. \end{aligned}$$

Interaction potential:

$$V_{\rm int}(\mathbf{x}) = 2\sum_{i<j} \Bigl(\sinh^{-2}x_{ij}^- + \sinh^{-2}x_{ij}^+\Bigr)\, a(a + S_{ij})\,.$$

Case 4. $P(z) = 4(1-z^2)$.
Change of variables: $z = \cos 2x$.
Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \big(\sin x_{ij}^- \, \sin x_{ij}^+\big)^a \prod_i (\sin x_i)^{\frac{1}{2}(1-\alpha-b)} (\cos x_i)^{\frac{1}{2}(1+\alpha-2\alpha_0-b)} \exp\Big(-\frac{\alpha_+}{2}\cos 2x_i\Big).$$

External potential:

$$\begin{aligned} U(x) = &-\alpha_+^2 \cos^2 2x + 2\alpha_+(b-\alpha_0)\cos 2x + 2(\alpha_+ + \alpha_-)(b+\alpha_0)\cos 2x \, \sin^{-2}2x \\ &+ \big((\alpha_+ + \alpha_-)^2 + (b+\alpha_0)^2 - 1\big)\sin^{-2}2x\,. \end{aligned}$$

Interaction potential:

$$V_{\rm int}(\mathbf{x}) = 2\sum_{i<j} \Big(\sin^{-2}x_{ij}^- + \sin^{-2}x_{ij}^+\Big)\, a(a+S_{ij})\,.$$

Case 5. $P(z) = 4z^2$.
Change of variables: $z = \mathrm{e}^{2x}$.
Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \sinh^a x_{ij}^- \prod_i \exp\Big[\frac{1}{2}\Big(\alpha_+ \mathrm{e}^{2x_i} - \alpha_- \mathrm{e}^{-2x_i}\Big) + (\alpha_0 - m)x_i\Big].$$

External potential:

$$U(x) = \alpha_+^2 \mathrm{e}^{4x} + 2\alpha_+(\alpha_0+b)\,\mathrm{e}^{2x} + 2\alpha_-(\alpha_0-b)\,\mathrm{e}^{-2x} + \alpha_-^2\,\mathrm{e}^{-4x}\,.$$

Interaction potential:

$$V_{\rm int}(\mathbf{x}) = 2\sum_{i<j} \sinh^{-2}x_{ij}^- \, a(a+S_{ij})\,.$$

Case 6. $P(z) = (1+z^2)^2$.
Change of variables: $z = \tan x$.
Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \sin^a x_{ij}^- \prod_i \cos^m x_i \, \exp\Big[(\alpha_+ + \alpha_-)x_i + \frac{1}{2}(\alpha_- - \alpha_+)\sin 2x_i - \frac{1}{2}\alpha_0 \cos 2x_i\Big].$$

External potential:

$$\begin{aligned} U(x) = &\frac{1}{2}\big((\alpha_+ - \alpha_-)^2 - \alpha_0^2\big)\cos 4x + 2\big(\alpha_-^2 - \alpha_+^2 + b\alpha_0\big)\cos 2x \\ &+ \alpha_0(\alpha_- - \alpha_+)\sin 4x + 2\big(\alpha_0(\alpha_+ + \alpha_-) + b(\alpha_+ - \alpha_-)\big)\sin 2x\,. \end{aligned}$$

Interaction potential:

$$V_{\rm int}(\mathbf{x}) = 2\sum_{i<j} \sin^{-2}x_{ij}^- \, a(a+S_{ij})\,.$$

Case 7. $P(z) = 4(1-z^2)(1-k^2z^2)$.
Change of variables: $z = \dfrac{\operatorname{cn} 2x}{\operatorname{dn} 2x}$.
Gauge factor:

$$\begin{aligned} \mu(\mathbf{x}) = \prod_{i<j} \left(\frac{\operatorname{sn} x_{ij}^- \operatorname{sn} x_{ij}^+}{1 - k^2 \operatorname{sn}^2 x_{ij}^- \operatorname{sn}^2 x_{ij}^+}\right)^a \prod_i \Big[&(\operatorname{sn} 2x_i)^{\frac{1}{2}\left(1-b-\frac{\alpha}{k'^2}\right)} (\operatorname{dn} 2x_i)^m \\ &\times \big(\operatorname{dn} 2x_i + \operatorname{cn} 2x_i\big)^{\frac{\alpha_+ + \alpha_-}{2k'^2}} \big(\operatorname{dn} 2x_i + k \operatorname{cn} 2x_i\big)^{-\frac{\alpha_+ + k^2\alpha_-}{2kk'^2}}\Big]\,. \end{aligned}$$

Here (and also in Cases 8 and 9) the functions $\operatorname{sn} x \equiv \operatorname{sn}(x|k)$, $\operatorname{cn} x \equiv \operatorname{cn}(x|k)$, and $\operatorname{dn} x \equiv \operatorname{dn}(x|k)$ are the usual Jacobian elliptic functions of *modulus* k, and $k' = \sqrt{1-k^2}$ is the *complementary modulus*.
External potential:

$$U(x) = A_7 \operatorname{sn}^2 2x + B_7 \operatorname{cn} 2x \operatorname{dn} 2x + \operatorname{sn}^{-2} 2x \left(C_7 + D_7 \operatorname{cn} 2x \operatorname{dn} 2x\right),$$

where

$$\begin{aligned}
A_7 &= k^2(b^2-1) + \frac{k^2\alpha_0}{k'^2}\Big(\frac{\alpha_0}{k'^2} - 2b\Big) + \frac{1}{k'^4}\,(\alpha_+ + k^2\alpha_-)^2\,,\\
B_7 &= \frac{2}{k'^2}\,(\alpha_+ + k^2\alpha_-)\Big(b - \frac{\alpha_0}{k'^2}\Big)\,,\\
C_7 &= b^2 - 1 + \frac{\alpha_0}{k'^2}\Big(\frac{\alpha_0}{k'^2} + 2b\Big) + \frac{1}{k'^4}\,(\alpha_+ + \alpha_-)^2\,,\\
D_7 &= \frac{2}{k'^2}\,(\alpha_+ + \alpha_-)\Big(b + \frac{\alpha_0}{k'^2}\Big)\,.
\end{aligned}$$

Interaction potential:

$$V_{\rm int}(\mathbf{x}) = 2\sum_{i<j}\left(\frac{\operatorname{cn}^2 x_{ij}^- \operatorname{dn}^2 x_{ij}^-}{\operatorname{sn}^2 x_{ij}^-} + \frac{\operatorname{cn}^2 x_{ij}^+ \operatorname{dn}^2 x_{ij}^+}{\operatorname{sn}^2 x_{ij}^+}\right) a(a+S_{ij})\,.$$

Case 8. $P(z) = 4(z^2-1)(1-k'^2 z^2)$.
Change of variables: $z = \dfrac{1}{\operatorname{dn} 2x}\,.$
Gauge factor:

$$\begin{aligned}
\mu(\mathbf{x}) = \prod_{i<j}\left(\frac{\operatorname{sn} x_{ij}^- \operatorname{sn} x_{ij}^+ \operatorname{cn} x_{ij}^- \operatorname{cn} x_{ij}^+}{1 - k^2 \operatorname{sn}^2 x_{ij}^- \operatorname{sn}^2 x_{ij}^+}\right)^a \prod_i \Bigg[(\operatorname{cn} 2x_i)^{\frac12\left[1-b-\frac{1}{k'k^2}(\alpha_+ + k'\alpha_0 + k'^2\alpha_-)\right]}\\
\times (\operatorname{sn} 2x_i)^{\frac12\left(1-b+\frac{\alpha}{k^2}\right)} (\operatorname{dn} 2x_i)^m \big(1+\operatorname{dn} 2x_i\big)^{-\frac{\alpha_+ + \alpha_-}{2k^2}} \big(k' + \operatorname{dn} 2x_i\big)^{\frac{\alpha_+ + k'^2\alpha_-}{2k'k^2}}\Bigg].
\end{aligned}$$

External potential:

$$U(x) = \operatorname{sn}^{-2} 2x\,(A_8 + B_8 \operatorname{dn} 2x) + \operatorname{cn}^{-2} 2x\,(C_8 + D_8 \operatorname{dn} 2x)\,,$$

where

$$\begin{aligned}
A_8 &= b^2 - 1 + \frac{\alpha_0}{k^2}\Big(\frac{\alpha_0}{k^2} - 2b\Big) + \frac{1}{k^4}\,(\alpha_+ + \alpha_-)^2\,,\\
B_8 &= \frac{2}{k^2}\,(\alpha_+ + \alpha_-)\Big(\frac{\alpha_0}{k^2} - b\Big)\,,\\
C_8 &= k'^2(b^2-1) + \frac{k'^2\alpha_0}{k^2}\Big(\frac{\alpha_0}{k^2} + 2b\Big) + \frac{1}{k^4}\,(\alpha_+ + k'^2\alpha_-)^2\,,\\
D_8 &= \frac{2}{k^2}\,(\alpha_+ + k'^2\alpha_-)\Big(\frac{\alpha_0}{k^2} + b\Big)\,.
\end{aligned}$$

Interaction potential:

$$V_{\rm int}(\mathbf{x}) = 2\sum_{i<j}\left(\frac{\operatorname{dn}^2 x_{ij}^-}{\operatorname{sn}^2 x_{ij}^- \operatorname{cn}^2 x_{ij}^-} + \frac{\operatorname{dn}^2 x_{ij}^+}{\operatorname{sn}^2 x_{ij}^+ \operatorname{cn}^2 x_{ij}^+}\right) a(a+S_{ij})\,.$$

Case 9. $P(z) = 4(1-z^2)(k'^2 + k^2 z^2)$.
Change of variables: $z = \operatorname{cn} 2x$.
Gauge factor:

$$\begin{aligned}
\mu(\mathbf{x}) = \prod_{i<j}\left(\frac{\operatorname{sn} x_{ij}^- \operatorname{sn} x_{ij}^+ \operatorname{dn} x_{ij}^- \operatorname{dn} x_{ij}^+}{1 - k^2 \operatorname{sn}^2 x_{ij}^- \operatorname{sn}^2 x_{ij}^+}\right)^a \prod_i \Bigg[(\operatorname{sn} 2x_i)^{\frac12(1-\alpha-b)} (\operatorname{dn} 2x_i)^{\frac12(1+\alpha_0-b)}\\
\times \big(1+\operatorname{cn} 2x_i\big)^{\frac12(\alpha_+ + \alpha_-)} \exp\Big[\frac{k^2\alpha_- - k'^2\alpha_+}{2kk'}\,\tan^{-1}\Big(\frac{k}{k'}\operatorname{cn} 2x_i\Big)\Big]\Bigg].
\end{aligned}$$

External potential:

$$U(x) = \mathrm{dn}^{-2}2x\,(A_9 + B_9\,\mathrm{cn}\,2x) + \mathrm{sn}^{-2}2x\,(C_9 + D_9\,\mathrm{cn}\,2x)\,,$$

where

$$\begin{aligned}
A_9 &= k'^2(1-b^2) + k'^2\alpha_0(2b-\alpha_0) + \frac{1}{k^2}\left(k'^2\alpha_+ - k^2\alpha_-\right)^2,\\
B_9 &= 2(b-\alpha_0)\big(k'^2\alpha_+ - k^2\alpha_-\big)\,,\\
C_9 &= (b+\alpha_0)^2 + (\alpha_+ + \alpha_-)^2 - 1\,,\\
D_9 &= 2(b+\alpha_0)(\alpha_+ + \alpha_-)\,.
\end{aligned}$$

Interaction potential:

$$V_{\mathrm{int}}(\mathbf{x}) = 2\sum_{i<j}\left(\frac{\mathrm{cn}^2 x_{ij}^-}{\mathrm{sn}^2 x_{ij}^-\,\mathrm{dn}^2 x_{ij}^-} + \frac{\mathrm{cn}^2 x_{ij}^+}{\mathrm{sn}^2 x_{ij}^+\,\mathrm{dn}^2 x_{ij}^+}\right)a(a+S_{ij})\,.$$

Remark 4. In Case 7, the alternative canonical form

$$P(z) = 4z^3 - g_2 z - g_3\,,\qquad g_2^3 > 27g_3^2\,,$$

leads to a spin generalization of the QES potential involving Weierstrass functions studied in [17]. The corresponding change of variables is $z = \wp(x+\omega_3)$, where $\wp(x) = \wp(x|g_2,g_3)$ is the Weierstrass function with invariants g_2, g_3, and $2\omega_3$ is its purely imaginary fundamental period. The gauge factor reads

$$\begin{aligned}
\mu(x) = \prod_{i<j}\big(\wp(x_i+\omega_3) - \wp(x_j+\omega_3)\big)^a \prod_i \Big[&\big(\wp'(x_i+\omega_3)\big)^\beta\\
&\times\big(\wp(x_i+\omega_3)-e_1\big)^{\gamma_1}\big(\wp(x_i+\omega_3)-e_2\big)^{\gamma_2}\big(\wp(x_i+\omega_3)-e_3\big)^{\gamma_3}\Big],
\end{aligned}$$

where

$$\begin{aligned}
\beta &= \frac12(1-b) + \frac{\alpha_+}{3}\,,\\
\gamma_j &= \frac{g_2\alpha_+ + 12(e_j\alpha_0 + \alpha_-)}{24(e_j-e_k)(e_j-e_l)}\,,\qquad (j,k,l) = \text{cyclic permutation of } (1,2,3)\,,
\end{aligned}$$

and e_j are the real (different) roots of $P(z)$. The external and interaction potentials are given by

$$U(x) = 4\beta(\beta-1)\wp(2x) + A\wp(x+\omega_3) + \big(B\wp^2(x+\omega_3) + C\wp(x+\omega_3) + D\big)\big(\wp'(x+\omega_3)\big)^{-2},$$

where

$$\begin{aligned}
A &= 2\alpha_0 + \frac49\,\alpha_+(2\alpha_+ + 3b)\,,\\
B &= 4\alpha_0^2 + \frac13\,(2\alpha_+ - 3b)(12\alpha_- + g_2\alpha_+)\,,\\
C &= 2\alpha_0\big(4\alpha_- + g_2(\alpha_+ - b)\big)\,,\\
D &= \Big(g_2 b + 4\alpha_- - \frac13\,g_2\alpha_+\Big)\Big(\alpha_- + \frac1{12}\,g_2\alpha_+\Big) + g_3\alpha_0(3b - 2\alpha_+)\,,
\end{aligned}$$

and

$$V_{\mathrm{int}}(\mathbf{x}) = 2\sum_{i<j}\big(\wp(x_{ij}^-) + \wp(x_{ij}^+)\big)\,a(a+S_{ij})\,.$$

The restriction of this spin model to the polynomial space $\mathcal{S}_m$ (see Remark 4) yields the scalar elliptic CS model in [17] provided $\alpha_0 = 0$ and $\alpha_- = -g_2\alpha_+/12$.

Remark 5. Case 1 with $\alpha_+ = 0$ yields the rational Calogero A_N spin model. Case 5 with $\alpha_+\alpha_- = 0$ is the model studied by Inozemtsev [25], while for $\alpha_+ = \alpha_- = 0$ the hyperbolic Sutherland A_N spin model is obtained. Case 6 with $\alpha_+ - \alpha_- = \alpha_0 = 0$ is the trigonometric Sutherland A_N spin model. The remaining potentials are new.

In Cases 1–5, the potential is ES if $\alpha_+ = 0$. In Case 5, the potential is also ES for $\alpha_- = 0$. The only ES potential in Case 6 is the trigonometric Sutherland potential ($\alpha_+ - \alpha_- = \alpha_0 = 0$). The remaining potentials, including all the elliptic potentials in Cases 7–9, are QES.

Remark 6. In order to qualify as a physical wavefunction, the spin functions $\psi(\mathbf{x})$ constructed in Section III must vanish at all points in which the corresponding potential $V(\mathbf{x})$ is singular. In addition, in Cases 1,2,3,5 the function ψ is required to be square-integrable over a suitable domain of $\mathbb{R}^N$. Both requirements impose certain constraints on the parameters α_ϵ and a defining the potential. A detailed analysis of the necessary and sufficient conditions on these parameters lies beyond the scope of this paper; see [18] for a complete solution of the analogous one-particle problem. However, it is not difficult in each case to provide sufficient conditions for the above requirements to hold. For example, in Case 5 the spin functions ψ vanish at the singularities of the potential if and only if $a > 0$, and are square-integrable over the domain $\{\mathbf{x} \in \mathbb{R}^N : x_1 > \cdots > x_N\}$ *provided* $\alpha_+ < 0$ and $\alpha_- > 0$.

Remark 7. In Cases 1–6, the gauge factor is of the form

$$\mu(\mathbf{x}) = \prod_{i<j} \left[f(x_{ij}^-)\, g(x_{ij}^+) \right]^a \prod_i h(x_i)\,, \tag{34}$$

with $g = 1$ or $g = f$. On the other hand, in the elliptic Cases 7–9 the gauge factor does not factorize as (34). This is consistent with the analogous result proved by Calogero for elliptic potentials in the scalar case [7].

V. CONCLUSIONS

We have developed in this paper a systematic method for constructing new families of exactly and quasi-exactly solvable spin Calogero–Sutherland models. The key idea consists in relating the physical spin Hamiltonian to a general quadratic combination involving the usual A_N-type Dunkl operators (1), (4), and the new family of Dunkl operators J_i^+ introduced in Section II (Eq. (8)). Our approach goes beyond the Lie algebraic method extensively used in the scalar case, since the Dunkl operators (8) do not span a Lie algebra. However, they are invariant under the projective action of $\mathrm{GL}(2,\mathbb{R})$, a fact that is exploited in Section IV to classify all the potentials obtained by this method up to translations.

The potentials constructed in this paper are all invariant under the A_N group consisting of permutations of the particles' coordinates and spins. A remarkable feature of the potentials in Cases 2–4 and 7–9 is their additional invariance under a change of sign of the spatial coordinate of any particle. Therefore, although these potentials are not invariant under the full B_N group of permutations and sign reversal of the particles' coordinates *and* spins, they are invariant under the restriction of the action of this group to the spatial coordinates. These models thus occupy an intermediate position between the usual spin CS models of A_N type and the fully B_N-invariant rational and trigonometric spin CS models introduced by Yamamoto [44]. In fact, while Dunkl has recently proved the exact solvability of the rational Yamamoto model [13], there are no exact results for the eigenfunctions of its trigonometric counterpart. It is to be expected that a suitable extension of the method developed in this paper to the B_N case will yield new families of (Q)ES spin CS models of B_N type, including the trigonometric Yamamoto model.

VI. APPENDIX

The formulae for the sums of the squares and the anticommutators of the Dunkl operators (8) are given by

$$\sum_i (J_i^-)^2 = \sum_i \partial_{z_i}^2 + 2a\sum_{i<j}\frac{1}{z_i - z_j}\left(\partial_{z_i} - \partial_{z_j}\right) - 2a\sum_{i<j}\frac{1}{(z_i - z_j)^2}\,(1 - K_{ij}), \tag{A1}$$

$$\begin{aligned}\sum_i (J_i^0)^2 = {}& \sum_i \left(z_i^2\partial_{z_i}^2 + (2-b)\,z_i\partial_{z_i}\right) + 2a\sum_{i<j}\frac{1}{z_i - z_j}\left(z_i^2\partial_{z_i} - z_j^2\partial_{z_j}\right)\\ & - a\sum_{i<j}\frac{z_i^2 + z_j^2}{(z_i - z_j)^2}\,(1 - K_{ij}) + a\sum_{i<j}(1 - K_{ij}) + \frac{a^2}{12}{\sum_{i,j,k}}'(1 - K_{ij}K_{jk}) + \frac{Nm^2}{4},\end{aligned} \tag{A2}$$

$$\begin{aligned}\sum_i (J_i^+)^2 = {}& \sum_j \left(z_i^4\partial_{z_i}^2 + 2(2-b)\,z_i^3\partial_{z_i}\right) + 2a\sum_{i<j}\frac{1}{z_i - z_j}\left(z_i^4\partial_{z_i} - z_j^4\partial_{z_j}\right)\\ & - a\sum_{i<j}\frac{z_i^4 + z_j^4}{(z_i - z_j)^2}\,(1 - K_{ij}) + a\sum_{i<j}(z_i + z_j)^2(1 - K_{ij})\\ & - 2am\sum_{i<j} z_i z_j + m(m-1)\sum_i z_i^2,\end{aligned} \tag{A3}$$

$$\begin{aligned}\frac{1}{2}\sum_i \{J_i^0,\ J_i^-\} = {}& \sum_i \left(z_i\partial_{z_i}^2 + \frac{1}{2}(2-b)\,\partial_{z_i}\right) + 2a\sum_{i<j}\frac{1}{z_i - z_j}\left(z_i\partial_{z_i} - z_j\partial_{z_j}\right)\\ & - a\sum_{i<j}\frac{z_i + z_j}{(z_i - z_j)^2}\,(1 - K_{ij}),\end{aligned} \tag{A4}$$

$$\begin{aligned}\frac{1}{2}\sum_i \{J_i^+, J_i^-\} = {}& \sum_i \left(z_i^2\partial_{z_i}^2 + (2-b)\,z_i\partial_{z_i}\right) + 2a\sum_{i<j}\frac{1}{z_i - z_j}\left(z_i^2\partial_{z_i} - z_j^2\partial_{z_j}\right)\\ & - a\sum_{i<j}\frac{z_i^2 + z_j^2}{(z_i - z_j)^2}\,(1 - K_{ij}) - \frac{a^2}{6}{\sum_{i,j,k}}'(1 - K_{ij}K_{jk})\\ & - \frac{mN}{2}\big(1 + a(N-1)\big),\end{aligned} \tag{A5}$$

$$\begin{aligned}\frac{1}{2}\sum_i \{J_i^0, J_i^+\} = {}& \sum_i \left(z_i^3\partial_{z_i}^2 + \frac{3}{2}(2-b)\,z_i^2\partial_{z_i}\right) + 2a\sum_{i<j}\frac{1}{z_i - z_j}\left(z_i^3\partial_{z_i} - z_j^3\partial_{z_j}\right)\\ & - a\sum_{i<j}\frac{z_i^3 + z_j^3}{(z_i - z_j)^2}\,(1 - K_{ij}) + a\sum_{i<j}(z_i + z_j)(1 - K_{ij})\\ & + \frac{m(2m-b)}{2}\sum_i z_i,\end{aligned} \tag{A6}$$

where

$$b = 1 + m + a(N-1).$$

Here the symbol $\sum'_{i,j,k}$ stands for summation in i, j, k with $i \neq j \neq k \neq i$. The following identities are needed for the computation of the potential:

$$\sum_{i,j,k}' \frac{1}{(z_i - z_j)(z_i - z_k)} = 0, \tag{A7}$$

$$\sum_{i,j,k}' \frac{z_i}{(z_i - z_j)(z_i - z_k)} = 0, \tag{A8}$$

$$\sum_{i,j,k}' \frac{z_i^2}{(z_i - z_j)(z_i - z_k)} = \frac{1}{3}N(N-1)(N-2), \tag{A9}$$

$$\sum_{i,j,k}' \frac{z_i^3}{(z_i - z_j)(z_i - z_k)} = (N-1)(N-2)\sum_i z_i, \tag{A10}$$

$$\sum_{i,j,k}' \frac{z_i^4}{(z_i - z_j)(z_i - z_k)} = (N-2)\sum_{i<j}(z_i + z_j)^2, \tag{A11}$$

$$\sum_{i \neq j} \frac{1}{z_i - z_j} = 0, \tag{A12}$$

$$\sum_{i \neq j} \frac{z_i}{z_i - z_j} = \frac{1}{2}N(N-1), \tag{A13}$$

$$\sum_{i \neq j} \frac{z_i^2}{z_i - z_j} = (N-1)\sum_i z_i, \tag{A14}$$

$$\sum_{i \neq j} \frac{z_i^3}{z_i - z_j} = (N-1)\sum_i z_i^2 + \sum_{i<j} z_i z_j. \tag{A15}$$

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